Improving the Jarrow-Yildirim Inflation Model

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1 Introduction

The most liquid inflation markets are those of the US, UK, France and Eurozone. Each is supported by a regular supply of government-issued inflation-linked bonds, and most trading desks provide swaps and vanilla options to some extent.

The Eurozone market is the most developed in terms of derivative products. Roughly in order from the most- to least-liquid, there are prices in zero-coupon (ZC) swaps, year-on-year (YoY) swaps and options, zero-coupon options; it is also possible to get prices for some first-order exotics such as YoY digitals and YoY range-accruals.

Within all markets the YoY vanilla option flows are concentrated on the zero-percent strike, with the YoY 0% floors being the most traded product, mainly because YoY swap trades with clients tend to have the coupons of the inflation leg floored at 0%. Other YoY strikes will trade less frequently, perhaps as hedges to structured products such as inflation-linked MTNs, and in periods where the market anticipates higher levels of inflation there may be increased amounts of trading of the higher-strike (eg 4% or 5%) caps.

All things considered, it is fair to say that smile modelling is less important for inflation markets than it is for the rates markets, say. In fact it is only the UK inflation market with its LPI product for pension funds which might present the need for a smile-enabled monte-carlo model, and even there one can find a reasonable workaround with approximation formulas.

In the author's experience the main priority for inflation modelling has been to produce a single model which the trading desk can use to:

- generate the convexity adjustments for YoY swap rates,
- calibrate to the term structure of YoY 0% strike vols,
- calibrate to the term structure of ZC 0% strike vols,
- generate payment-delay adjustments for products like pay-as-you-go swaps.

The benefit of having a single model to generate these volatility-dependent prices or price adjustments is twofold: trading and risk-control teams prefer to have a single model which can satisfactorily explain the prices seen in the market, and secondly it offers the trading desk more hedging strategies such as hedging ZC options with YoY or hedging the YoY swap adjustments with YoY options.

It goes without saying that a single model which achieves all these requirements can be used as the basis for pricing and risk managing the few path-dependent products that get requested from time to time – even though it is not smile enabled we can be confident that it correctly reflects the core volatility and correlation levels of the market (traders tend to see the spread between ZC and YoY option volatilities as reflecting the correlations amongst the term structure of YoY options).

The earliest arbitrage-free model for inflation was presented in an article by Jarrow and Yildirim, and is based on the FX analogy. Since it is in fact nothing other than the well known HJM cross-currency model, it was easy for trading houses to code up a new inflation wrapper for the FX model and *voila!* the JY model became the standard approach for dealing with inflation-linked derivative products.

In recent years the JY model has seen its popularity fade, as market-model approaches have become developed, and it is easy to understand why: the JY model seems to suffer from over-parametrization, it diffuses the rather esoteric real-yield process, it is not obvious how to calibrate. In contrast, the market models take a more intuitive slimmed-down approach and diffuse the inflation process directly, and it tends to be obvious how they should be calibrated.

The fact is however that with a small amount of work, the JY model can be modified to produce a new model that is perfectly able to acheive all of the requirements listed above. In this article we present the mathematics behind a re-factoring of the JY model which produces a very satisfactory inflation model. Furthermore, we show that a marginally reduced version of this new model can be implemented very quickly as a wrapper around an existing implementation of the JY model, which means that you can have this better version up and running with a minimal amount of effort.

2 The JY model

The JY model is based on *real* and *nominal* economies, each with its own yield curve, which are connected by a spot process for the inflation index. The inflation index is analogous to the FX spot process and dictates the current nominal price of real assets.

The JY model specifies the dynamics of the real and nominal discount factors and the spot index process (respectively $B_r(t;T)$, $B_n(t;T)$ and $I(t)$), under the risk-neutral measure $\mathbb{P}_n$, as follows:

$$\begin{aligned}
\frac{\mathrm{d}B_n(t;T)}{B_n(t;T)} &= r_n(t)\,\mathrm{d}t + \sigma_{B_n}(t;T)\,\mathrm{d}W_t^n, \\
\frac{\mathrm{d}B_r(t;T)}{B_r(t;T)} &= \left[r_r(t) - \sigma_I(t)\sigma_{B_r}(t;T)\rho_{rI}(t)\right]\,\mathrm{d}t + \sigma_{B_r}(t;T)\,\mathrm{d}W_t^r, \\
\frac{\mathrm{d}I(t)}{I(t)} &= \left[r_n(t) - r_r(t)\right]\,\mathrm{d}t + \sigma_I(t)\,\mathrm{d}W_t^I,
\end{aligned}$$

where $\left(W_t^n, W_t^r, W_t^I\right)$ is a Brownian motion under $\mathbb{P}_n$ with correlation matrix

$$\begin{pmatrix} 1 & \rho_{nr} & \rho_{nI} \\ \rho_{nr} & 1 & \rho_{rI} \\ \rho_{nI} & \rho_{rI} & 1 \end{pmatrix}$$

Gaussian dynamics for the rates are specified:

$$\sigma_{B_k}(t;T) = \sigma_k(t) \int_t^T e^{-\int_t^s \lambda_k(u)\,\mathrm{d}u}\,\mathrm{d}s, \quad k = n, r$$

with σ_n, σ_r, λ_n and λ_r being deterministic functions (the short-rate vols and the mean reversions).

It is not obvious how to fully calibrate the JY model, but below is an outline of the approach this author found to be most practicable and useful (particularly because it gives good-quality risks):

1. calibrate the term structure of nominal volatilites, the σ_n, in order to correctly price libor caps at a given strike (depending on the nominal vol hedge to be used),

2. calibrate the term structure of real volatilites, the σ_r, in order to produce the correct convexity adjustment for YoY swap rates,

3. calibrate the term structure of CPI volatilies, the σ_I, in order to correctly price a chosen set of YoY options (usually being the 0% floors).

This calibration recipe still leaves a number of parameters unconstrained: the two mean reversions λ_n and λ_r and the three correlations ρ_{nI}, ρ_{rI} and ρ_{nr}. It is possible to use historical series to put a figure on ρ_{nr} and in principle the same applies to ρ_{nI} and ρ_{rI} but estimation is rather more difficult because of the low number of historical index data points. Instead it was preferred to set $\rho_{nI} = \rho_{rI} = 0$ because it means that the step where σ_I is calibrated to YoY options will not disturb the previous calibration to the convexity adjustments (which depend on σ_r, σ_n, ρ_{nI} and ρ_{rI}). For the mean reversions a pragmatic solution works best, a compromise based on historical analysis and mathematical simplicity, which for the Eurozone market meant choosing $\lambda_n = \lambda_r = 10\%$.

3 Refactoring the JY model

The key to improving the calibration and the dynamics of the JY model is to not diffuse the real-yield curve but instead to diffuse an inflation curve, as we show in this section. To motivate the new terms we introduce, we recall that in the JY model the time-t value of the inflation index with maturity T is given by:

$$I(t;T) = I(t)\frac{B_r(t;T)}{B_n(t;T)} = I(t)\, e^{\int_t^T f_n(t;s)-f_r(t;s)\,\mathrm{d}s},$$

where $f_n(t;s)$ and $f_r(t;s)$ represent the time-t values of the instantaneous forward rates with maturity s in the nominal and real economies. Clearly the spread $f_n(t;s) - f_r(t;s)$ defines an implied curve of instantaneous inflation forward rates, which we write as $f_i(t;s)$ and which our new model diffuses (rather than $f_r(t;s)$ in the JY model).

To this end we define a process $ZC(t;T)$ which represents the continuously-compounded inflationary growth between times t and T:

$$ZC(t;T) := e^{\int_t^T f_i(t;s)\,\mathrm{d}s}.$$

The new model keeps the same processes as JY for the nominal discount factors and the inflation spot index, but replaces the process of real discount factors with the ZC process. The specification is:

$$\begin{aligned}
\frac{\mathrm{d}B_n(t;T)}{B_n(t;T)} &= r_n(t)\,\mathrm{d}t + \sigma_{B_n}(t;T)\,\mathrm{d}W_t^n, \\
\frac{\mathrm{d}ZC(t;T)}{ZC(t;T)} &= r_i(t)\,\mathrm{d}t + \mu(t;T)\,\mathrm{d}t + \sigma_{ZC}(t;T)\,\mathrm{d}W_t^i, \\
\frac{\mathrm{d}I(t)}{I(t)} &= r_i(t)\,\mathrm{d}t + \sigma_I(t)\,\mathrm{d}W_t^I,
\end{aligned}$$

where it can be checked that for no-arbitrage the drift term must be given as

$$\mu(t;T) = \sigma_{ZC}(t;T)^2 - \rho_{iI}\sigma_I(t)\sigma_{ZC}(t;T) + \left[\rho_{nI}\sigma_I(t) - \rho_{ni}\sigma_{ZC}(t;T)\right]\sigma_{B_n}(t;T)$$

and where $\left(W_t^n, W_t^i, W_t^I\right)$ is a Brownian motion under $\mathbb{P}_n$ with correlation matrix

$$\begin{pmatrix} 1 & \rho_{ni} & \rho_{nI} \\ \rho_{ni} & 1 & \rho_{iI} \\ \rho_{nI} & \rho_{iI} & 1 \end{pmatrix}$$

Gaussian dynamics for the nominal and inflation rates are specified:

$$\sigma_{B_n}(t;T) = \sigma_n(t) \int_t^T e^{-\int_t^s \lambda_n(u)\, \mathrm{d}u}\, \mathrm{d}s,$$
$$\sigma_{ZC}(t;T) = \sigma_i(t) \int_t^T e^{-\int_t^s \lambda_i(u)\, \mathrm{d}u}\, \mathrm{d}s,$$

with σ_n, σ_i, λ_n and λ_i being deterministic functions (the short-rate vols and the mean reversions).

In this new model we have

$$I(t;T) = I(t)ZC(t;T),$$

from which it follows that the forward-index terms $I(t;T)$ have dynamics given by:

$$\frac{\mathrm{d}I(t;T)}{I(t;T)} = -\left[\rho_{nI}\sigma_I(t) - \rho_{ni}\sigma_{ZC}(t;T)\right] \mathrm{d}t + \sigma_I(t)\,\mathrm{d}W^I(t) + \sigma_{ZC}(t;T)\,\mathrm{d}W^i(t),$$

and therefore the Black-Scholes volatitilies of inflation options (both YoY and ZC) are only dependent on the parameters σ_I and σ_i and the correlation ρ_{iI}.

The convexity adjutment is given by:

$$\mathbb{E}\left(\frac{I(T_2)}{I(T_1)}\right) = \frac{I(t;T_2)}{I(t;T_1)} \times \frac{\exp\left(\int_t^{T_1} \left[\rho_{iI}\sigma_I(s) - \sigma_{ZC}(s;T_1)\right]\left[\sigma_{ZC}(s;T_2) - \sigma_{ZC}(s;T_1)\right] \mathrm{d}s\right)}{\exp\left(\int_t^{T_1} \left[\rho_{nI}\sigma_I(s) - \rho_{ni}\sigma_{ZC}(s;T_1)\right]\left[\sigma_{B_n}(s;T_2) - \sigma_{B_n}(s;T_1)\right] \mathrm{d}s\right)}$$

The convexity adjustment terms factor neatly into two parts: one depending only on the inflation parameters, and the other which also depends on the nominal vol terms. This second term arises from there being a payment delay on the denominator term.

4 Calibration of the new model

The new specification gives a much better relationship between the parameters of the model and the prices of the assets that are traded in the market and this immediately improves the prospects for a better calibration. The correspondence is:

- the prices of nominal options (eg libor caps or floors) are determined by σ_n and λ_n (as they are in the JY model),
- the Black-Scholes volatilities of inflation options are determined by σ_I, σ_i, ρ_{iI} and λ_i,
- the YoY convexity adjustments can be tweaked with ρ_{nI} and ρ_{ni}.

Importantly, the way that σ_i and σ_I affect the Black-Scholes volatilities of ZC and YoY options is quite different, so we now have a mechanism for adjusting the spread between ZC and YoY options: if we put more inflation volatility into the model with σ_i we will tend to increase the ZC-YoY volatility spread, whereas if we use σ_I to increase the inflation volatility we will tend to decrease the ZC-YoY volatility spread.

Taking all this into consideration, the following scheme for calibration of this new model has been found to work very well in practice – it will fit the YoY volatilities exactly and has been able to generate a good quality fit to ZC volatilities and YoY convexity adjustments.

1. start the calibration loop with $\lambda_i = 0.1$, $\rho_{ni} = 0$, $\rho_{iI} = 0$ and $\rho_{nI} = 0$, and $\sigma_i = 0.005$,

2. calibrate the term structure of nominal volatilites, the σ_n, and adjust the level of mean reversion λ_n in order to correctly price the nominal hedge instruments (eg libor caps and swaptions at a given strike),

3. calibrate the term structure of the σ_I, in order to correctly hit the Black-Scholes volatilities of the YoY options at a given strike (again, depending on the inflation vol hedge to be used),

4. increase (decrease) σ_i in order to get generally higher (lower) levels of BS volatilities for the inflation ZC options, and return to step 3

5. increase (decrease) λ_i in order to put more (less) curvature into the shape of the BS volatilities of the ZC options, and return to step 3.

6. increase (decrease) ρ_{ni} in order to widen (narrow) the convexity adjustment, and return to step 3

Steps 4, 5 and 6 can obviously be encapsulated in a minimization routine. The two remaining parameters which we have not yet addressed, ρ_{iI} and ρ_{nI} have a more subtle effect on the shapes of the calibrated instruments, but within the calibration loop we are suggesting here they can be used to change the way market movements in the YoY option volatilities generate moves in the ZC volatilities and in the convexity adjustments. In other words they give some degree of control to the trader to choose how market moves in YoY get carried across into the ZC and convexity markets.

5 A quick implementation of (a slightly-restricted version of) the new model

At the cost of losing one degree of freedom, it is possible to map the new model back onto the JY model. This allows us to obtain an almost immediate implementation of the new scheme within an implementation of JY. Namely, we insist that we must always have $\lambda_n = \lambda_i$ and therefore lose the flexibility to calibrate to nominal swaptions as well as caps, for example; this is a reasonable compromise.

In other words, if we have a set $\{\lambda_n, \lambda_i; \sigma_n(t), \sigma_i(t), \sigma_I(t), \rho_{ni}(t), \rho_{nI}(t), \rho_{iI}(t)\}$ of mean reversions and term-structure values for the parameters of the new model and furthermore have $\lambda_n = \lambda_i$, then with the following definitions:

$$\begin{aligned}
\lambda_r &:= \lambda_i, \\
\sigma_r(t) &:= \sqrt{\sigma_n(t)^2 + \sigma_i(t)^2 - 2\rho_{ni}(t)\sigma_n(t)\sigma_i(t)}, \\
\rho_{nr}(t) &:= \frac{1}{\sigma_r(t)}\left(\sigma_n(t) - \rho_{ni}(t)\sigma_i(t)\right), \\
\rho_{rI}(t) &:= \frac{1}{\sigma_r(t)}\left(\rho_{nI}(t)\sigma_n(t) - \rho_{iI}(t)\sigma_i(t)\right),
\end{aligned}$$

we have another set $\{\lambda_n, \lambda_r; \sigma_n(t), \sigma_r(t), \sigma_I(t), \rho_{nr}(t), \rho_{nI}(t), \rho_{rI}(t)\}$ of mean reversions and term-structure parameters which we can use in a JY model to generate exactly the same volatility distributions.

This means that by writing a simple wrapper at the front and back of an existing implementation of JY, we can very quickly build an implementation of this improved model. A little more work on a basic spreadsheet calibration routine will then be enough to have a workable model which the trading desk can experiment with. None of the internals of the pricing engines needs to be re-plumbed.

6 Better PnL Explain

In the JY model a bump on the nominal vol parameter σ_n will impact the valuation of a book of inflation options potentially in two ways:

1. it will affect Black-Scholes volatility,
2. it will affect the convexity adjustment.

The new model is in a much better situation since a bump in the nominal vol will affext only the convexity adjustment, and there it only causes a change through the payment delay component of the adjustment; in the JY model the change in the convexity adjustment will be due both to the change of payment delay and the change in implied inflation volatility.